

TOTALLY NULL SURFACES IN NEUTRAL KÄHLER 4-MANIFOLDS

NIKOS GEORGIOU, BRENDAN GUILFOYLE, AND WILHELM KLINGENBERG

ABSTRACT. We study the totally null surfaces of the neutral Kähler metric on certain 4-manifolds. The tangent spaces of totally null surfaces are either self-dual (α -planes) or anti-self-dual (β -planes) and so we consider α -surfaces and β -surfaces. The metric of the examples we study, which include the spaces of oriented geodesics of 3-manifolds of constant curvature, are anti-self-dual, and so it is well-known that the α -planes are integrable and α -surfaces exist. These are holomorphic Lagrangian surfaces, which for the geodesic spaces correspond to totally umbilic foliations of the underlying 3-manifold.

The β -surfaces are less known and our interest is mainly in their description. In particular, we classify the β -surfaces of the neutral Kähler metric on TN , the tangent bundle to a Riemannian 2-manifold N . These include the spaces of oriented geodesics in Euclidean and Lorentz 3-space, for which we show that the β -surfaces are affine tangent bundles to curves of constant geodesic curvature on S^2 and H^2 , respectively. In addition, we construct the β -surfaces of the space of oriented geodesics of hyperbolic 3-space.

1. INTRODUCTION

Neutral Kähler 4-manifolds exhibit remarkably different behaviour than their positive-definite counterparts. The failure of the complex structure J to tame the symplectic structure Ω means that 2-planes in the tangent space of a point can be both holomorphic and Lagrangian. Under favorable conditions (namely the vanishing of the self-dual conformal curvature) such planes are integrable and there exist holomorphic Lagrangian surfaces.

In the space $\mathbb{L}(M)$ of oriented geodesics of a 3-manifold of constant curvature M (on which a natural neutral Kähler structure exists) such surfaces play a distinctive role: they correspond to totally umbilic foliations of M [3] [5] [6].

Holomorphic Lagrangian planes are totally null, that is, the induced metric identically vanishes on the plane. Moreover, with respect to the Hodge star operator of the neutral metric, the self-dual 2-forms vanish on these planes. There exists however another class of totally null planes, upon which the anti-self-dual forms vanish. The former planes are referred to as α -planes, while the latter are β -planes.

In this note we consider the β -surfaces in certain neutral Kähler 4-manifolds, which include spaces $\mathbb{L}(M)$ of oriented geodesics of 3-manifolds M of constant curvature. In the cases of $M = \mathbb{E}^3, \mathbb{E}_1^3, \mathbb{H}^3$ we compute the β -surfaces explicitly and show that they include $\mathbb{L}(\mathbb{E}^2), \mathbb{L}(\mathbb{H}^2)$.

In particular, we prove that

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Main Theorem:

A β -surface in $\mathbb{L}(\mathbb{E}^3)$ is an affine tangent bundle over a curve of constant geodesic curvature in (S^2, g_{rnd}) .

A β -surface in $\mathbb{L}(\mathbb{E}_1^3)$ is an affine tangent bundle over a curve of constant geodesic curvature in $(H^2 g_{\text{hyp}})$.

A β -surface in $\mathbb{L}(\mathbb{H}^3)$ is a piece of a torus which, up to isometry, is either

- (1) $\mathbb{L}(\mathbb{H}^2)$, where $\mathbb{H}^2 \subset \mathbb{H}^3$, or
- (2) $\mathcal{C}_1 \times \mathcal{C}_2 \subset S^2 \times S^2 - \bar{\Delta}$, where the \mathcal{C}_1 is a circle given by the intersection of the 2-sphere and a plane containing the north pole, and \mathcal{C}_2 is the image of \mathcal{C}_1 under reflection in the horizontal plane through the origin.

In the next section we discuss self-duality for planes in neutral Kähler 4-manifolds and their properties. We then turn to the neutral metric on TN and the special case $\mathbb{L}(\mathbb{E}^3)$ and $\mathbb{L}(\mathbb{E}_1^3)$. In the final section we characterize the β -surfaces in $\mathbb{L}(\mathbb{H}^3)$.

2. NEUTRAL METRICS ON 4-MANIFOLDS

2.1. Self-dual and anti-self-dual 2-forms. Consider the neutral metric G on \mathbb{R}^4 given in standard coordinates (x^1, x^2, x^3, x^4) by

$$ds^2 = (dx^1)^2 + (dx^2)^2 - (dx^3)^2 - (dx^4)^2.$$

Throughout, we denote \mathbb{R}^4 endowed with this metric by $\mathbb{R}^{2,2}$.

The space of 2-forms on $\mathbb{R}^{2,2}$ is a 6-dimensional linear space that splits naturally with respect to the Hodge star operator $*$ of G into two 3-dimensional spaces: $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$, the space of self-dual and anti-self-dual 2-forms. Thus, if $\omega \in \Lambda^2$, then $\omega = \omega_+ + \omega_-$, where $*\omega_+ = \omega_+$ and $*\omega_- = -\omega_-$.

We can easily find a basis for Λ_+^2 and Λ_-^2 . First, define the *double null* basis of 1-forms:

$$\Theta^1 = dx^1 + dx^3, \quad \Theta^2 = dx^2 - dx^4, \quad \Theta^3 = -dx^2 - dx^4, \quad \Theta^4 = dx^1 - dx^3,$$

so that the metric is

$$ds^2 = \Theta^1 \otimes \Theta^4 - \Theta^2 \otimes \Theta^3.$$

Proposition 1. *If $\omega \in \Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$, with $\omega = \omega_+ + \omega_-$, then*

$$\omega_+ = a_1 \Theta^1 \wedge \Theta^2 + b_1 \Theta^3 \wedge \Theta^4 + c_1 (\Theta^1 \wedge \Theta^4 - \Theta^2 \wedge \Theta^3),$$

$$\omega_- = a_2 \Theta^1 \wedge \Theta^3 + b_2 \Theta^2 \wedge \Theta^4 + c_2 (\Theta^1 \wedge \Theta^4 + \Theta^2 \wedge \Theta^3),$$

for $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$.

Proof. This follows from computing the Hodge star operator acting on 2-forms:

$$*(\Theta^1 \wedge \Theta^4) = -\Theta^2 \wedge \Theta^3, \quad *(\Theta^2 \wedge \Theta^4) = -\Theta^2 \wedge \Theta^4, \quad *(\Theta^1 \wedge \Theta^3) = -\Theta^1 \wedge \Theta^3,$$

$$*(\Theta^3 \wedge \Theta^4) = \Theta^3 \wedge \Theta^4, \quad *(\Theta^1 \wedge \Theta^2) = \Theta^1 \wedge \Theta^2.$$

□

2.2. Totally Null Planes.

Definition 1. A plane $P \subset \mathbb{R}^{2,2}$ is *totally null* if every vector in P is null with respect to G , and the inner product of any two vectors in P is zero.

A plane P is *self-dual* if $\omega_+(P) = 0$ for all $\omega_+ \in \Lambda_+^2$, and *anti-self-dual* if $\omega_-(P) = 0$ for all $\omega_- \in \Lambda_-^2$. Self-dual planes are also called α -planes, while anti-self-dual planes are called β -planes.

Proposition 2. A plane P is totally null iff P is either self-dual or anti-self-dual.

Proof. Suppose all self-dual forms vanish on P and let $\{V, W\}$ be a basis for P . Let (e_1, e_2, e_3, e_4) be the vector basis of $\mathbb{R}^{2,2}$ that is dual to $(\Theta^1, \Theta^2, \Theta^3, \Theta^4)$ and $V = V^j e_j$, $W = W^j e_j$. Since all of the self-dual 2-forms vanish on P , we have from the expression of ω_+ in Proposition 1 that

$$V^1 W^2 = W^1 V^2 \quad V^3 W^4 = W^3 V^4, \quad (2.1)$$

$$V^1 W^4 - V^2 W^3 = W^1 V^4 - W^2 V^3. \quad (2.2)$$

We can assume without loss of generality that V and W are orthogonal: $G(V, W) = 0$, which in frame components says that

$$V^1 W^4 + W^1 V^4 = V^2 W^3 + W^2 V^3.$$

Combining this with equation (2.2) we have that

$$V^1 W^4 = V^2 W^3 \quad W^1 V^4 = W^2 V^3. \quad (2.3)$$

Multiplying the first equation of (2.3) by W^1 we have

$$V^1 W^4 W^1 = V^2 W^3 W^1,$$

which, by virtue of the first equation of (2.1), is

$$V^1 W^4 W^1 = W^2 W^3 V^1.$$

Thus

$$G(W, W)V^1 = 2(W^1 W^4 - W^2 W^3)V^1 = 0.$$

Similarly, multiplying the first equation of (2.3) by W^2 , and the second equation by W^3 and W^4 , applying equations (2.1), we find that

$$G(W, W)V^2 = G(W, W)V^3 = G(W, W)V^4 = 0.$$

Thus, either $G(W, W) = 0$ or $V = 0$. Since the latter is not true, we conclude that W is a null vector.

On the other hand, multiplying the second equation of (2.3) by V^1 and V^2 , and the first by V^3 and V^4 , utilizing equations (2.1), we have

$$G(V, V)W^1 = G(V, V)W^2 = G(V, V)W^3 = G(V, V)W^4 = 0.$$

Thus V is also a null vector, and the plane spanned by V and W is totally null, as claimed.

An analogous argument establishes that a plane on which all anti-self-dual 2-forms vanish is totally null.

Conversely, suppose that a plane P is totally null. That is, in terms of a vector basis V and W as before

$$V^1 V^4 = V^2 V^3 \quad W^1 W^4 = W^2 W^3, \quad (2.4)$$

$$V^1W^4 + V^4W^1 - V^2W^3 - V^3W^2 = 0. \quad (2.5)$$

Multiplying equation (2.5) by V^1, V^3, W^1 and W^3 , yields, with the aid of equations (2.4):

$$V^2(V^3W^1 - V^1W^3) = V^1(V^3W^2 - V^1W^4), \quad (2.6)$$

$$V^4(V^3W^1 - V^1W^3) = V^3(V^3W^2 - V^1W^4), \quad (2.7)$$

$$W^2(V^1W^3 - V^3W^1) = W^1(V^2W^3 - V^4W^1), \quad (2.8)$$

$$W^4(V^1W^3 - V^3W^1) = W^3(V^2W^3 - V^4W^1). \quad (2.9)$$

Now, adding V^1 times equation (2.8), W^1 times equation (2.6), V^3 times equation (2.9) and W^3 times equation (2.7) and using equation (2.5), we obtain

$$(V^1W^2 - V^2W^1 + V^3W^4 - V^4W^3)(V^1W^3 - V^3W^1) = 0. \quad (2.10)$$

By a similar manipulation we find that

$$(V^1W^2 - V^2W^1 + V^3W^4 - V^4W^3)(V^2W^4 - V^4W^2) = 0. \quad (2.11)$$

Now suppose that P , in addition to being totally null, is Lagrangian. If $J(V)$ is not in P , then, since $G(W, J(V)) = \Omega(W, V) = 0$, the metric would be identically zero on the 3-space spanned by $\{V, W, J(V)\}$. For a non-degenerate metric G on $\mathbb{R}^{2,2}$ this is not possible. Thus $J(V) \in P$ and so P is a complex plane. It follows easily that P is self-dual.

On the other hand, suppose that the totally null plane P is not Lagrangian. Then $\Omega(V, W) \neq 0$ or

$$V^1W^2 - V^2W^1 + V^3W^4 - V^4W^3 \neq 0.$$

By equations (2.10) and (2.11), we have $V^1W^3 - V^3W^1 = V^2W^4 - V^4W^2 = 0$. Moreover, substituting these in (2.6) to (2.9) we conclude that $V^1W^4 - V^4W^1 + V^2W^3 - V^3W^2 = 0$. Then, by Proposition 1 we must have $\omega_-(V, W) = 0$, which completes the result. \square

2.3. Kähler Structure on $\mathbb{R}^{2,2}$. Up to an overall sign, there are two complex structures on $\mathbb{R}^{2,2}$ that are compatible with the metric G :

$$J(X^1, X^2, X^3, X^4) = (-X^2, X^1, -X^4, X^3),$$

and

$$J'(X^1, X^2, X^3, X^4) = (-X^2, X^1, X^4, -X^3).$$

By compatibility we mean that $G(J \cdot, J \cdot) = G(\cdot, \cdot)$, and similarly for J' .

We can utilize these and define two symplectic forms by $\Omega = G(\cdot, J \cdot)$ and $\Omega' = G(\cdot, J' \cdot)$. That is

$$\Omega = dx^1 \wedge dx^2 - dx^3 \wedge dx^4 \quad \Omega' = dx^1 \wedge dx^2 + dx^3 \wedge dx^4.$$

Thus, the symplectic 2-form Ω is self-dual while Ω' is anti-self-dual.

Moreover

Proposition 3. *An α -plane is holomorphic and Lagrangian with respect to (J, Ω) , while a β -plane is holomorphic and Lagrangian with respect to (J', Ω') .*

Proof. The proof follows from arguments similar to those of Proposition 2. \square

Given a null vector V in $\mathbb{R}^{2,2}$, the planes spanned by $\{V, J(V)\}$ and $\{V, J'(V)\}$ are easily seen to be totally null. More explicitly, the set of totally null planes is, in fact, $S^1 \cup S^1$, which can be parameterized as follows. For $a, b \in \mathbb{R}$, $\phi \in [0, 2\pi)$ and $\epsilon = \pm 1$, consider the vector in $\mathbb{R}^{2,2}$ given by

$$V_\phi^\epsilon(a, b) = (a \cos \phi + b \sin \phi, a \sin \phi - b \cos \phi, a, -\epsilon b).$$

Let P_ϕ^ϵ be the plane containing $V_\phi^\epsilon(a, b)$ as a and b vary over \mathbb{R} . Then a quick check shows that P_ϕ^+ is self-dual, while P_ϕ^- is anti-self-dual.

2.4. Neutral Kähler 4-manifolds. Let (M, G, J, Ω) be a smooth neutral Kähler 4-manifold. Thus M is a smooth 4-manifold, G is a neutral metric, while J is a complex structure that is compatible with G and $\Omega(\cdot, \cdot) = G(J\cdot, \cdot)$ is a closed non-degenerate (symplectic) 2-form.

The existence of a unitary frame at a point of M implies that it is possible to apply the algebra of the last section pointwise on M , and we therefore have $S^1 \cup S^1$ worth of totally null planes at each point. On a compact 4-manifold, the existence of an oriented 2-dimensional distribution implies topological restrictions on M [7], and so not every compact 4-manifold admits a neutral Kähler structure. However, the examples we consider are non-compact and the neutral Kähler structure will be given explicitly.

On any (pseudo)-Riemannian 4-manifold (M, G) the Riemann curvature tensor can be considered as an endomorphism of $\Lambda^2(M)$. The splitting $\Lambda^2(M) = \Lambda_+^2(M) \oplus \Lambda_-^2(M)$ with respect to the Hodge star operator $*$ yields a block decomposition of the Riemann curvature tensor

$$\text{Riem} = \begin{bmatrix} \text{Weyl}^+ + \frac{1}{12}R & \text{Ric} \\ \text{Ric} & \text{Weyl}^- + \frac{1}{12}R \end{bmatrix},$$

where Ric is the Ricci tensor, R is the scalar curvature and Weyl^\pm are the self- and anti-self-dual Weyl curvature tensors [1].

Definition 2. A (pseudo)-Riemannian 4-manifold (M, G) is *anti-self-dual* if the self-dual part of the Weyl conformal curvature tensor vanishes: $\text{Weyl}^+ = 0$.

A well-known result of Penrose states:

Theorem 1. [2][9] *The α -surfaces of a neutral Kähler 4-manifold (M, G) are integrable iff (M, G) is anti-self-dual.*

3. NEUTRAL KÄHLER METRIC ON TN

Let (N, g) be a Riemannian 2-manifold and consider the total space TN of the tangent bundle to N . Choose conformal coordinates ξ on N so that $ds^2 = e^{2u} d\xi d\bar{\xi}$

for some function $u = u(\xi, \bar{\xi})$, and the corresponding complex coordinates (ξ, η) on TN obtained by identifying

$$(\xi, \eta) \leftrightarrow \eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} \in T_\xi N.$$

The coordinates (ξ, η) define a natural complex structure on TN by

$$\mathbb{J} \left(\frac{\partial}{\partial \xi} \right) = i \frac{\partial}{\partial \xi} \quad \mathbb{J} \left(\frac{\partial}{\partial \eta} \right) = i \frac{\partial}{\partial \eta}.$$

In [5] a neutral Kähler structure was introduced on TN . In the above coordinate system, the symplectic 2-form is

$$\Omega = 2e^{2u} \operatorname{Re} (d\eta \wedge d\bar{\xi} + 2\eta \partial_\xi u \, d\xi \wedge d\bar{\xi}), \quad (3.1)$$

while the neutral metric \mathbb{G} is

$$\mathbb{G} = 2e^{2u} \operatorname{Im} (d\bar{\eta} d\xi - 2\eta \partial_\xi u \, d\xi d\bar{\xi}). \quad (3.2)$$

Here we have introduced the notation ∂_ξ for differentiation with respect to ξ .

Note 1. When $u = 0$ we retrieve the neutral Kähler metric on $\mathbb{R}^4 = \mathbb{T}\mathbb{R}^2$, where

$$\xi = \frac{1}{2} [x^1 + x^3 + i(x^2 + x^4)] \quad \eta = \frac{1}{2} [x^2 - x^4 + i(-x^1 + x^3)],$$

or

$$\begin{aligned} x^1 &= \frac{1}{2} [\xi + \bar{\xi} + i(\eta - \bar{\eta})] & x^2 &= \frac{1}{2} [-i(\xi - \bar{\xi}) + \eta + \bar{\eta}], \\ x^3 &= \frac{1}{2} [\xi + \bar{\xi} - i(\eta - \bar{\eta})] & x^4 &= \frac{1}{2} [-i(\xi - \bar{\xi}) - \eta - \bar{\eta}]. \end{aligned}$$

Proposition 4. *The double null basis for (TN, \mathbb{G}) is*

$$\begin{aligned} \Theta^1 &= 2\operatorname{Re}(d\xi) & \Theta^2 &= 2e^{2u} \operatorname{Re}(d\eta + 2\eta \partial_\xi u \, d\xi), \\ \Theta^3 &= 2\operatorname{Im}(d\xi) & \Theta^4 &= 2e^{2u} \operatorname{Im}(d\eta + 2\eta \partial_\xi u \, d\xi). \end{aligned}$$

Proof. A straight-forward check shows that

$$ds^2 = \Theta^1 \otimes \Theta^4 - \Theta^2 \otimes \Theta^3,$$

as claimed. \square

The coordinate expressions for self-dual and anti-self-dual 2-forms on TN are

Proposition 5. *If $\omega \in \Lambda^2(TN) = \Lambda_+^2(TN) \oplus \Lambda_-^2(TN)$, with $\omega = \omega_+ + \omega_-$, then*

$$\begin{aligned} \omega_+ &= a_1(d\xi \wedge d\eta + d\bar{\xi} \wedge d\bar{\eta}) + b_1[d\xi \wedge d\bar{\eta} + d\bar{\xi} \wedge d\eta + 2(\bar{\eta} \partial_\xi u - \eta \partial_\xi u) d\xi \wedge d\bar{\xi}] \\ &\quad + ic_1(d\xi \wedge d\eta - d\bar{\xi} \wedge d\bar{\eta}), \end{aligned}$$

$$\begin{aligned} \omega_- &= ia_2 d\xi \wedge d\bar{\xi} + ib_2[d\xi \wedge d\bar{\eta} - d\bar{\xi} \wedge d\eta + 2(\bar{\eta} \partial_\xi u + \eta \partial_\xi u) d\xi \wedge d\bar{\xi}] \\ &\quad + ic_2(d\eta \wedge d\bar{\eta} + 2\eta \partial_\xi u d\xi \wedge d\bar{\eta} + 2\bar{\eta} \partial_\xi u d\bar{\xi} \wedge d\eta + 4\eta \bar{\eta} \partial_\xi u \partial_\xi u d\xi \wedge d\bar{\xi}), \end{aligned}$$

for $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$.

3.1. α -surfaces in TN . We first note that

Proposition 6. *The neutral Kähler metric \mathbb{G} on TN is anti-self-dual.*

Proof. A calculation using the coordinate expression (3.2) of the metric shows that the only non-vanishing component of the conformal curvature tensor is

$$W_{\xi\bar{\xi}}^{\eta\bar{\eta}} = i(\eta\partial_{\xi}\kappa - \bar{\eta}\partial_{\bar{\xi}}\kappa),$$

where κ is the Gauss curvature of (N, g) .

Thus, from Proposition 5, for any $\omega_+ \in \Lambda_+^2(TN)$, $W(\omega_+) = 0$. That is, the metric is anti-self-dual. \square

By applying Theorem 1 we have

Corollary 1. *There exists α -surfaces, i.e. holomorphic Lagrangian surfaces, in (TN, \mathbb{J}, Ω) .*

3.2. β -surfaces in TN .

Proposition 7. *An immersed surface $\Sigma \subset TN$ is a β -surface iff locally it is given by $(s, t) \rightarrow (\xi(s, t), \eta(s, t))$ where*

$$\xi = se^{iC_0} + \xi_0 \quad \eta = (te^{iC_0} + \eta_0)e^{-2u},$$

for $C_0 \in \mathbb{R}$ and $\xi_0, \eta_0 \in \mathbb{C}$.

Proof. By Proposition 5 surface $f : \Sigma \rightarrow TN$ is a β -surface iff

$$f^*(d\xi \wedge d\bar{\xi}) = 0 \quad f^*(d(\eta e^{2u}) \wedge d(\bar{\eta} e^{2u})) = 0, \quad (3.3)$$

and

$$f^*(d\xi \wedge d(\bar{\eta} e^{2u}) - d\bar{\xi} \wedge d(\eta e^{2u})) = 0. \quad (3.4)$$

The first equation of (3.3) implies that the map $(s, t) \rightarrow \xi(s, t)$ is not of maximal rank, and as it cannot be of rank zero (as this would mean that Σ is a fibre of $\pi : TN \rightarrow N$, and is therefore an α -surface) it must be of rank 1. By the implicit function theorem either

$$\xi(s, t) = \xi(s, t(s)) \quad \text{or} \quad \xi(s, t) = \xi(s(t), t).$$

Without loss of generality, we will assume the former: $\xi = \xi(s)$.

Similarly, the second equation of (3.3) implies that either

$$\eta e^{2u} = \psi(s, t) = \psi(s, t(s)) \quad \text{or} \quad \eta e^{2u} = \psi(s, t) = \psi(s(t), t).$$

Here, we must have the latter $\eta e^{2u} = \psi(t)$, or else the surface Σ would be singular.

Turning now to equation of (3.4), we have

$$\frac{d\xi}{ds} \frac{d\bar{\psi}}{dt} = \frac{d\bar{\xi}}{ds} \frac{d\psi}{dt}.$$

By separation of variables we see that

$$\frac{d\xi}{ds} = e^{2iC_0} \frac{d\bar{\xi}}{ds} \quad \frac{d\psi}{ds} = e^{2iC_0} \frac{d\bar{\psi}}{ds},$$

for some real constant C_0 . These can be integrated to

$$\xi = h_1(s)e^{iC_0} + \xi_0 \quad \eta = (h_2(t)e^{iC_0} + \eta_0)e^{-2u},$$

for complex constants ξ_0 and η_0 and real functions h_1 and h_2 of s and t , respectively.

Finally, we can reparameterize s and t so that $h_1 = s$ and $h_2 = t$, as claimed. \square

3.3. The Oriented Geodesic Spaces TS^2 and TH^2 . In the cases where $N = S^2$ or $N = H^2$ endowed with a metric of constant Gauss curvature ($e^{2u} = 4(1 \pm \xi\bar{\xi})^{-2}$), the above construction yields the neutral Kähler metric on the space $\mathbb{L}(\mathbb{E}^3)$ of oriented affine lines or on the space $\mathbb{L}(\mathbb{E}_1^3)$ of future-pointing time-like lines, in \mathbb{E}^3 or \mathbb{E}_1^3 (respectively) [6].

In what follows we consider only the Euclidean case, although analogous results hold for the Lorentz case. We define the map Φ which sends $\mathbb{L}(\mathbb{E}^3) \times \mathbb{R}$ to \mathbb{E}^3 as follows: Φ takes an oriented line γ and a real number r to that point in \mathbb{E}^3 which lies on γ and is an affine parameter distance r from the point on γ closest to the origin.

Proposition 8. [5] *The map can be written as $\Phi((\xi, \eta), r) = (z, t) \in \mathbb{C} \oplus \mathbb{R} = \mathbb{E}^3$ where the local coordinate expressions are:*

$$z = \frac{2(\eta - \bar{\eta}\xi^2) + 2\xi(1 + \xi\bar{\xi})r}{(1 + \xi\bar{\xi})^2} \quad t = \frac{-2(\eta\bar{\xi} + \bar{\eta}\xi) + (1 - \xi^2\bar{\xi}^2)r}{(1 + \xi\bar{\xi})^2}, \quad (3.5)$$

and

$$\eta = \frac{1}{2}(z - 2t\xi - \bar{z}\xi^2) \quad r = \frac{\bar{\xi}z + \xi\bar{z} + (1 - \xi\bar{\xi})t}{1 + \xi\bar{\xi}}. \quad (3.6)$$

For α -surfaces, we have

Proposition 9. *A holomorphic Lagrangian surface in TS^2 corresponds to the oriented normals to totally umbilic surfaces in \mathbb{E}^3 i.e. round spheres or planes.*

On the other hand:

Proposition 10. *A β -surface in TS^2 is an affine tangent bundle over a curve of constant geodesic curvature in (S^2, g_{rnd}) .*

Proof. By Proposition 7, the β -surfaces are given by

$$\xi = se^{iC_0} + \xi_0 \quad \eta = (1 + \xi\bar{\xi})^2(te^{iC_0} + \eta_0).$$

Clearly this is a real line bundle over a curve on S^2 . By a rotation this can be simplified to

$$\xi = s + \xi_0 e^{-iC_0} \quad \eta = (1 + \xi\bar{\xi})^2(t + \eta_0 e^{-iC_0}),$$

and after an affine reparameterization of s and t we can set

$$\xi = s + iC_1 \quad \eta = (1 + \xi\bar{\xi})^2(t + iC_2).$$

Projecting onto S^2 we get the curve $\xi = s + iC_1$ with unit tangent \vec{T} and normal vector \vec{N} (with respect to the round metric)

$$\vec{T} = \frac{(1 + \xi\bar{\xi})}{2\sqrt{2}} \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \bar{\xi}} \right) \quad \vec{N} = \frac{i(1 + \xi\bar{\xi})}{2\sqrt{2}} \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \bar{\xi}} \right).$$

Considered as a set of vectors on S^2 , the β -surface is

$$\begin{aligned}\eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} &= (1 + \xi \bar{\xi})^2 (t + iC_2) \frac{\partial}{\partial \xi} + (1 + \xi \bar{\xi})^2 (t - iC_2) \frac{\partial}{\partial \bar{\xi}} \\ &= 2\sqrt{2}(1 + \xi \bar{\xi})(t\vec{T} + C_2\vec{N}).\end{aligned}$$

These form a real line bundle over the base curve - which do not pass through the origin in the fibre of TS^2 for $C_2 \neq 0$. For $C_2 = 0$, this is exactly the tangent bundle to the curve.

The geodesic curvature of this curve is

$$\begin{aligned}g(\vec{N}, \nabla_{\vec{T}} \vec{T}) &= N_k T^j (\partial_j T^k + \Gamma_{jl}^k T^l) \\ &= N_k T^j \partial_j T^k + N^k T^j T^l (2\partial_j g_{lk} - \partial_k g_{jl}) \\ &= \sqrt{2}C_1.\end{aligned}$$

□

A similar calculation establishes:

Proposition 11. *A β -surface in TH^2 is an affine tangent bundle over a curve of constant geodesic curvature in (H^2, g_{hyp}) .*

We also have the following:

Corollary 2. *Given an affine plane P in \mathbb{E}^3 , the set $\mathbb{L}(\mathbb{E}^2)$ of oriented lines contained in P is a β -surface in TS^2 .*

Proof. By Proposition 7, the β -surfaces are given by

$$\xi = se^{iC_0} + \xi_0 \quad \eta = (1 + \xi \bar{\xi})^2 (te^{iC_0} + \eta_0)$$

Isometries of \mathbb{E}^3 induce isometries on TS^2 and hence preserves β -surfaces. Thus we can translate and rotate P so that it is vertical and contains the t -axis. Thus we can consider the β -surface Σ with $\xi_0 = \eta_0 = 0$, and then using the map Φ we find the two parameter family of oriented lines in \mathbb{E}^3 to be

$$z = \frac{2[(1 - s^4)t + sr]}{1 + s^2} e^{iC_0} \quad t = \frac{-4s(1 + s^2)t + (1 - s^2)r}{1 + s^2}$$

This is a vertical plane containing the t -axis, and Σ consists of all the oriented lines in this plane. □

4. ORIENTED GEODESICS IN HYPERBOLIC 3-SPACE

We briefly recall the basic construction of the canonical neutral Kähler metric on the space $\mathbb{L}(\mathbb{H}^3)$ of oriented geodesics of \mathbb{H}^3 - further details can be found in [3].

Consider the 4-manifold $\mathbb{P}^1 \times \mathbb{P}^1$ endowed with the canonical complex structure $\mathbb{J} = j \oplus j$ and complex coordinates μ_1 and μ_2 . If we let $\bar{\Delta} = \{(\mu_1, \mu_2) : \mu_1 \bar{\mu}_2 = -1\}$ then $\mathbb{L}(\mathbb{H}^3) = \mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$. We introduce the neutral Kähler metric and symplectic form on $\mathbb{L}(\mathbb{H}^3)$ by

$$\mathbb{G} = -i \left[\frac{1}{(1 + \mu_1 \bar{\mu}_2)^2} d\mu_1 \otimes d\bar{\mu}_2 - \frac{1}{(1 + \bar{\mu}_1 \mu_2)^2} d\bar{\mu}_1 \otimes d\mu_2 \right], \quad (4.1)$$

and

$$\Omega = - \left[\frac{1}{(1 + \mu_1 \bar{\mu}_2)^2} d\mu_1 \wedge d\bar{\mu}_2 + \frac{1}{(1 + \bar{\mu}_1 \mu_2)^2} d\bar{\mu}_1 \wedge d\mu_2 \right]. \quad (4.2)$$

Proposition 12. *A double null basis for $(\mathbb{L}(\mathbb{H}^3), \mathbb{G})$ is*

$$\begin{aligned} \Theta^1 &= \mathbb{R}e \left(\frac{d\mu_1}{1 + \mu_1 \bar{\mu}_2} - \frac{d\mu_2}{1 + \bar{\mu}_1 \mu_2} \right) & \Theta^2 &= \mathbb{R}e \left(\frac{d\mu_1}{1 + \mu_1 \bar{\mu}_2} + \frac{d\mu_2}{1 + \bar{\mu}_1 \mu_2} \right), \\ \Theta^3 &= -\mathbb{I}m \left(\frac{d\mu_1}{1 + \mu_1 \bar{\mu}_2} - \frac{d\mu_2}{1 + \bar{\mu}_1 \mu_2} \right) & \Theta^4 &= -\mathbb{I}m \left(\frac{d\mu_1}{1 + \mu_1 \bar{\mu}_2} + \frac{d\mu_2}{1 + \bar{\mu}_1 \mu_2} \right). \end{aligned}$$

Proof. A straight-forward computation shows that

$$ds^2 = \Theta^1 \otimes \Theta^4 - \Theta^2 \otimes \Theta^3,$$

as claimed \square

The coordinate expressions for self-dual and anti-self-dual 2 forms on $\mathbb{L}(\mathbb{H}^3)$ are easily found to be:

Proposition 13. *If $\omega \in \Lambda^2(\mathbb{L}(\mathbb{H}^3)) = \Lambda_+^2(\mathbb{L}(\mathbb{H}^3)) \oplus \Lambda_-^2(\mathbb{L}(\mathbb{H}^3))$, with $\omega = \omega_+ + \omega_-$, then*

$$\begin{aligned} \omega_+ &= (a_1 + ic_1) \frac{d\mu_1 \wedge d\mu_2}{|1 + \bar{\mu}_1 \mu_2|^2} + (a_1 - ic_1) \frac{d\bar{\mu}_1 \wedge d\bar{\mu}_2}{|1 + \bar{\mu}_1 \mu_2|^2} + b_1 \left[\frac{d\mu_1 \wedge d\bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} + \frac{d\bar{\mu}_1 \wedge d\mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} \right], \\ \omega_- &= -i(a_2 + c_2) \frac{d\mu_1 \wedge d\bar{\mu}_1}{|1 + \bar{\mu}_1 \mu_2|^2} - i(a_2 - c_2) \frac{d\mu_2 \wedge d\bar{\mu}_2}{|1 + \bar{\mu}_1 \mu_2|^2} + ib_2 \left[\frac{d\mu_1 \wedge d\bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} - \frac{d\bar{\mu}_1 \wedge d\mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} \right]. \end{aligned}$$

4.1. α -surfaces in $\mathbb{L}(\mathbb{H}^3)$. Once again, the neutral metric on $\mathbb{L}(\mathbb{H}^3)$ is anti-self-dual, indeed, it is conformally flat, and so there exists α -surfaces in $\mathbb{L}(\mathbb{H}^3)$. These are found to be the normal congruence to the totally umbilic surfaces in \mathbb{H}^3 :

Proposition 14. [4] *A smooth surface Σ in $\mathbb{L}(\mathbb{H}^3)$ is totally null iff Σ is the oriented normal congruence of*

- (1) *a geodesic sphere, or*
- (2) *a horosphere, or*
- (3) *a totally geodesic surface*

in \mathbb{H}^3 .

4.2. β -surfaces in $\mathbb{L}(\mathbb{H}^3)$.

Proposition 15. *Let Σ be a β -surface in $\mathbb{L}(\mathbb{H}^3)$. Then Σ is a piece of a torus which, up to isometry, is either*

- (1) *$\mathbb{L}(\mathbb{H}^2)$, where $\mathbb{H}^2 \subset \mathbb{H}^3$, or*
- (2) *$\mathcal{C}_1 \times \mathcal{C}_2 \subset S^2 \times S^2 - \bar{\Delta}$, where the \mathcal{C}_1 is a circle given by the intersection of the 2-sphere and a plane containing the north pole, and \mathcal{C}_2 is the image of \mathcal{C}_1 under reflection in the horizontal plane through the origin.*

Proof. Let $f : \Sigma \rightarrow \mathbb{L}(\mathbb{H}^3)$ be an immersed β -surface. Then for every anti-self-dual 2-form ω_- we have $f^*\omega_- = 0$. Then we obtain the following equations

$$f^*(d\mu_1 \wedge d\bar{\mu}_1) = 0 \quad f^*(d\mu_2 \wedge d\bar{\mu}_2) = 0, \quad (4.3)$$

$$f^* \left(\frac{d\mu_1 \wedge d\bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} - \frac{d\bar{\mu}_1 \wedge d\mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} \right) = 0. \quad (4.4)$$

The first equation of (4.3) implies that the map $(u, v) \mapsto \mu_1(u, v)$ is not of maximal rank and since it cannot be of rank zero (otherwise Σ would be an α -surface) it must be of rank 1. By the implicit function theorem either

$$\mu_1(u, v) = \mu_1(u, v(u)) \quad \text{or} \quad \mu_1(u, v) = \mu_1(u(v), v).$$

Without loss of generality, we will assume the former: $\mu_1 = \mu_1(u)$.

Similarly, the second equation of (4.3) implies that

$$\mu_2(u, v) = \mu_2(u, v(u)) \quad \text{or} \quad \mu_2(u, v) = \mu_2(u(v), v).$$

Here, we must have $\mu_2 = \mu_2(v)$, or else the surface Σ would be singular.

The equation (4.4) yields

$$\ln \mu_2 - \ln \bar{\mu}_2 + \ln(1 + \bar{\mu}_1 \mu_2) - \ln(1 + \mu_1 \bar{\mu}_2) = h_1(u) + h_2(v), \quad (4.5)$$

$$\ln \bar{\mu}_1 - \ln \mu_1 + \ln(1 + \bar{\mu}_1 \mu_2) - \ln(1 + \mu_1 \bar{\mu}_2) = w_1(u) + w_2(v), \quad (4.6)$$

for some complex functions h_1, h_2, w_1, w_2 .

If $h_i = a_i e^{i\phi_i}$ for $i = 1, 2$, where $a_1 = a_1(u)$, $\phi_1 = \phi_1(u)$ and $a_2 = a_2(v)$, $\phi_2 = \phi_2(v)$ are real functions, we obtain

$$h_1(u) = ia_1 \quad h_2(v) = ia_2.$$

By a similar argument, there are real functions $b_1 = b_1(u)$ and $b_2 = b_2(v)$ such that (4.5) and (4.6) become

$$\ln \mu_2 - \ln \bar{\mu}_2 + \ln(1 + \bar{\mu}_1 \mu_2) - \ln(1 + \mu_1 \bar{\mu}_2) = i(a_1(u) + a_2(v)), \quad (4.7)$$

$$\ln \bar{\mu}_1 - \ln \mu_1 + \ln(1 + \bar{\mu}_1 \mu_2) - \ln(1 + \mu_1 \bar{\mu}_2) = i(b_1(u) + b_2(v)). \quad (4.8)$$

Finally from combining equations (4.7) and (4.8) we have

$$\ln \left(\frac{1 + \bar{\mu}_1 \mu_2}{1 + \mu_1 \bar{\mu}_2} \right) = -2i(f(u) + g(v)).$$

We are thus led to consider the curves $\mathcal{C}_1, \mathcal{C}_2$ on S^2 given locally by non-constant functions $\mu_1 : \mathbb{R} \rightarrow S^2 : u \mapsto \mu_1(u)$ and $\mu_2 : \mathbb{R} \rightarrow S^2 : v \mapsto \mu_2(v)$ which satisfy

$$1 + \mu_1 \bar{\mu}_2 = (1 + \bar{\mu}_1 \mu_2) e^{2i(f+g)},$$

for $f = f(u)$ and $g = g(v)$.

If we switch to polar coordinates $\mu_1 = \lambda_1(u) e^{i\theta_1(u)}$ and $\mu_2 = \lambda_2(v) e^{i\theta_2(v)}$, this reduces to

$$\sin[f(u) + g(v)] = \lambda_1(u) \lambda_2(v) \sin[\theta_1(u) - f(u) - \theta_2(v) - g(v)]. \quad (4.9)$$

By a rotation we can set μ_2 to zero for some $v = v_0$, that is, $\lambda_2(v_0) = 0$. We find from equation (4.9) that

$$\sin[f(u) + g(v_0)] = 0,$$

and so letting $g_0 = g(v_0)$, we conclude that $f = -g_0$. Putting this back into (4.9) we have

$$\sin[g(v) - g_0] = \lambda_1(u)\lambda_2(v)\sin[\theta_1(u) - \theta_2(v) - g(v) + g_0]. \quad (4.10)$$

Thus for a fixed $u = u_0$ we have

$$\lambda_1(u_0)\lambda_2(v)\sin[\theta_1(u_0) - \theta_2(v) - g(v) + g_0] = \lambda_1(u)\lambda_2(v)\sin[\theta_1(u) - \theta_2(v) - g(v) + g_0],$$

or, for $v \neq v_0$

$$\lambda_1(u_0)\sin[\theta_1(u_0) - \theta_2(v) - g(v) + g_0] = \lambda_1(u)\sin[\theta_1(u) - \theta_2(v) - g(v) + g_0]. \quad (4.11)$$

Differentiating this relationship with respect to v yields

$$\begin{aligned} \lambda_1(u_0)\cos[\theta_1(u_0) - \theta_2(v) - g(v) + g_0] \partial_v(\theta_2 + g) \\ = \lambda_1(u)\cos[\theta_1(u) - \theta_2(v) - g(v) + g_0] \partial_v(\theta_2 + g). \end{aligned} \quad (4.12)$$

If $\partial_v(\theta_2 + g) \neq 0$, then we can cancel this factor and square both sides of equations (4.11) and (4.12) to find that $\lambda_1 = \lambda_1(u_0)$. However, from the functional relation in equation (4.10), this means that θ_1 is also constant. Thus μ_1 would be constant, which is not true.

We conclude that $\partial_v(\theta_2 + g) = 0$, or equivalently, $g(v) = -\theta_2(v) + g_1$. Substituting this back into equation (4.10) we have

$$\sin[\theta_2(v) + C_0] = \lambda_1(u)\lambda_2(v)\sin[\theta_1(u) + C_0],$$

where $C_0 = g_0 - g_1$.

One solution of this equation is $\theta_1 = \theta_2 = -C_0$, which is the case $\Sigma = \mathbb{L}(\mathbb{H}^2)$, where $\mathbb{H}^2 \subset \mathbb{H}^3$. Otherwise, we can separate variables

$$\frac{\sin[\theta_2(v) + C_0]}{\lambda_2(v)} = \lambda_1(u)\sin[\theta_1(u) + C_0] = C_1 \neq 0.$$

This yields

$$\mu_1 = \frac{C_1 e^{i\theta_1(u)}}{\sin[\theta_1(u) + C_0]} \quad \mu_2 = \frac{\sin[\theta_2(v) + C_0] e^{i\theta_2(v)}}{C_1}.$$

By a rotation of S^2 we can set C_0 to zero, and with a natural choice of parameterization of the curves, the final form is

$$\mu_1 = \frac{C_1 e^{iu}}{\sin u} \quad \mu_2 = \frac{\sin v e^{iv}}{C_1},$$

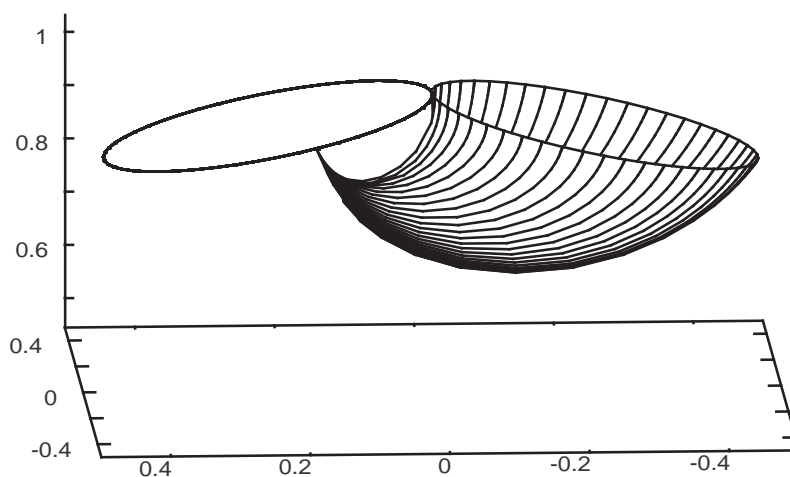
for $u, v \in [0, 2\pi)$.

These are the tori of part (2) in the statement. To see that they are circles note that if we view S^2 in \mathbb{R}^3 given by

$$x = \frac{\mu + \bar{\mu}}{1 + \mu\bar{\mu}} \quad y = \frac{-i(\mu - \bar{\mu})}{1 + \mu\bar{\mu}} \quad z = \frac{1 - \mu\bar{\mu}}{1 + \mu\bar{\mu}},$$

then the first curve parameterizes the intersection of S^2 with the plane $y + C_1(z - 1) = 0$, while the second is the intersection with the plane $y - C_1(z + 1) = 0$. \square

In the ball model of \mathbb{H}^3 these 2-parameter families of geodesics can be visualized as the set of geodesics that begin on a circle on the boundary and end on another circle of the same radius on the boundary, the two circles having a single point of intersection, as illustrated below.



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NIKOS GEORGIU, DEPARTMENT OF COMPUTING AND MATHEMATICS, INSTITUTE OF TECHNOLOGY, TRALEE, CLASH TRALEE, CO. KERRY, IRELAND.

E-mail address: `nikos.georgiou@research.ittralee.ie`

BRENDAN GUILFOYLE, DEPARTMENT OF COMPUTING AND MATHEMATICS, INSTITUTE OF TECHNOLOGY, TRALEE, CLASH, TRALEE, CO. KERRY, IRELAND.

E-mail address: `brendan.guilfoyle@ittralee.ie`

WILHELM KLINGENBERG, DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DURHAM, DURHAM DH1 3LE, UNITED KINGDOM.

E-mail address: `wilhelm.klingenberg@durham.ac.uk`